*Discipline:* **Physics**

*Subject:* **Electromagnetic Theory**

*Unit 27:*

*Lesson/ Module:* **Radiation by Point Sources-I**


*Author (CW):* **Prof. V. K. Gupta**

*Department/ University:* **Department of Physics and Astrophysics,**

- **University of Delhi, New Delhi-110007**

# Contents

***Learning Objectives:***

***From this module, a continuation of module 26, students may get to know about the following:***

1. *The calculation of radiation by moving sources from the electromagnetic fields obtained in the previous module.*
2. *Radiation and its angular distribution in the nonrelativistic case – Larmor formula for emitted radiation.*
3. *Total radiation in the relativistic case obtained from the generalization of the Larmor formula.*
4. *Radiation losses in charged particle accelerators – linear and circular accelerators.*
5. *Angular distribution of radiation in the relativistic case. Study of two special cases –acceleration parallel to or perpendicular to velocity and their comparison.*

# 27. Radiation by Point Charges

## *27.1 Introduction*

As we have emphasized again and again, one of our main tasks is, given a source, to find the fields and ultimately the radiation produced by it. The sources can be localized charges and currents, like waveguides, the input from which is fed to the antennas to produce radiation. Alternately we have a charged particle, or more likely a beam of charged particle moving along a given trajectory and possibly emitting radiation during its motion. This is the case that we are considering now. We have a particle having charge *e*, moving along a given but arbitrary trajectory $\vec{r}(t)$ with velocity $\vec{v}(t) = c\vec{\beta}$. The electric field $\vec{E}(\vec{x},t)$ and magnetic induction $\vec{B}(\vec{x},t)$ produced by the charge are given by

$$\vec{E}(\vec{x},t) = \frac{e}{4\pi\varepsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}.\hat{n})^3 R^2}\right]_{ret} + \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}.\hat{n})^3 R}\right]_{ret} \tag{35}$$

$$\vec{B} = \frac{1}{c}[\hat{n}\times\vec{E}]_{ret} \tag{36}$$

Here $\gamma$ is the Lorentz boost factor $1/\sqrt{1-v^2/c^2}$, $\vec{R} = \vec{x} - \vec{r}(t) = R\hat{n}$ and "*ret*" after the square bracket, $[\ ]_{ret}$, implies that the entire expression is to be evaluated at the retarded time $\tau_0$ given by $[x - r(\tau_0)]^2 = 0$ and the retardation requirement $x_0 > r_0(\tau_0)$, $\tau$ being the proper time of the particle.

An accelerated charged particle produces radiation. Radiation is the irreversible flow of electromagnetic energy from the source to infinity. This is possible only because electromagnetic fields associated with accelerating particles fall off as $1/r$ rather than as $1/r^2$ as in the case of a charge at rest or moving uniformly. So the total energy flux obtained from the Poynting vector is finite at infinity.

## *27.2 Larmor Formula*

Let us now find the radiation produced by such a charged particle. The first term in $\vec{E}(\vec{x},t)$, *the velocity field*, falls off as ~$1/R^2$ and does not produce radiation. It is the second term which is responsible for radiation. Let us first calculate the radiation produced by a particle which is accelerating but is observed from a frame of reference in which its velocity is small: $\beta<<1$. In such a frame of reference the acceleration field $\vec{E}_a$ is given by

$$\vec{E}_a(\vec{x},t) = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R}\right]_{ret} = \frac{e}{4\pi\varepsilon_0 c}\left[\frac{(\hat{n}.\dot{\vec{\beta}})\hat{n}-\dot{\vec{\beta}}}{R}\right]_{ret} \tag{3}$$

$$\vec{B}_a(\vec{x},t) = \frac{1}{c}\hat{n}\times\vec{E}_a = \frac{e}{4\pi\varepsilon_0 c^2}\left[\frac{\dot{\vec{\beta}}\times\hat{n}}{R}\right]_{ret} = \frac{e\mu_0}{4\pi}\left[\frac{\dot{\vec{\beta}}\times\hat{n}}{R}\right]_{ret} \tag{4}$$

The instantaneous energy flux is given by the Pointing vector

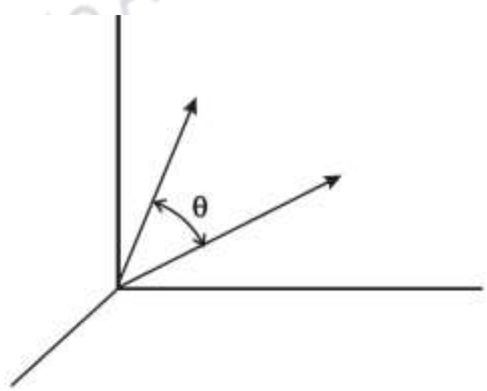
$$\vec{S} = \vec{E}\times\vec{B}/\mu_0 = \frac{1}{\mu_0 c}\vec{E}\times(\hat{n}\times\vec{E}) = \frac{1}{\mu_0 c}\left|\vec{E}_a\right|^2\hat{n} \tag{5}$$

This is the power radiated per unit area along the radius vector. The radially outward power output per unit solid angle is

$$\frac{dP}{d\Omega} = R^2\vec{S}.\hat{n} = \frac{1}{\mu_0 c}\left|R\vec{E}_a\right|^2 = \frac{e^2}{(4\pi\varepsilon_0)(4\pi c)}\left|\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})\right|^2 \tag{6}$$

on substituting for $\vec{E}_a$ from equation (3). If $\Theta$ is the angle between the acceleration $\dot{\vec{v}}(=c\dot{\vec{\beta}})$ and $\hat{n}$ **[See Figure 14.3 from Jackson Edition 2]** then the power radiated can be written as

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi\varepsilon_0)(4\pi c^3)}\left|\dot{\vec{v}}\right|^2\sin^2\Theta\,. \tag{7}$$




This is the characteristic $\sin^2\Theta$ dependence, which is also the characteristic of dipole radiation. In fact if you look carefully it is just the dipole result in disguise. Since, the direction of the electric field is taken to be the direction of polarization, from equation (3) we see that the radiation is polarized in the plane containing $\dot{\vec{v}}$ and $\hat{n}$. The total instantaneous power radiated is obtained by integrating equation (7) over all solid angle:

$$P=\int\frac{dP}{d\Omega}d\Omega=\frac{e^2}{(4\pi\varepsilon_0)(4\pi c^3)}\left|\dot{\vec{v}}\right|^2\int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi\sin^2\theta=\frac{e^2}{6\pi\varepsilon_0 c^3}\left|\dot{\vec{v}}\right|^2 \qquad (8)$$

This is the so called *Larmor Formula* for power radiated by a nonrelativistic accelerated point charge. The power radiated is independent of the sign of charge as well as whether the particle is accelerating or decelerating.

- Obviously a charged particle moving in a circular orbit even with a constant speed is accelerating all the time and hence must radiate. Classically this applies to an atom as well, in which case the electron moving in an orbit around the nucleus should continue to radiate, lose energy and eventually fall into the nucleus. Of course nothing of the kind happens; the fact that we exist is proof of that. This was one of the major paradoxes of classical physics which was ultimately resolved with the advent of quantum mechanics. But that is altogether another story.

### *27.3 Relativistic Generalization of Larmor Formula*

Larmor formula (8) for total power radiated can be generalized to arbitrary velocities by arguments about covariance under Lorentz transformations. Radiated electromagnetic energy, like mechanical energy is the zeroth component of a four vector. Similarly time is the zeroth component of position four-vector. It seems plausible therefore, and it can be shown more rigorously, that their ratio, the power radiated, $P=\frac{dW}{dt}$ is a Lorentz invariant quantity. If we can find a *unique* Lorentz invariant which reduces to Larmor formula in the limit, then we have our desired generalization of the formula. There are many Lorentz invariants that reduce to the Larmor formula in the nonrelativistic limit, but there are some other constraints that have to be met. From expressions (1) and (2) for the fields it is clear that the required formula must involve only $\vec{\beta}$ and $\dot{\vec{\beta}}$. Thus in the covariant form the four vectors that can be involved are $P^\mu$ and $\frac{dP^\mu}{d\tau}$. The only scalars we can form out of these are

$$P^\mu P_\mu, P^\mu\frac{dP_\mu}{d\tau} \text{ and } \frac{dP^\mu}{d\tau}\frac{dP_\mu}{d\tau}.$$

Now $P^\mu P_\mu=m^2c^2$, where $m$ is the mass of the particle, is just a constant. $P^\mu\frac{dP_\mu}{d\tau}$ reduces to zero in the nonrelativistic limit. We are thus left with only $\frac{dP^\mu}{d\tau}\frac{dP_\mu}{d\tau}$. The covariant generalization is therefore some multiple of $\frac{dP^\mu}{d\tau}\frac{dP_\mu}{d\tau}$. Writing Larmor formula as

$$P=\frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}\right) \qquad (9)$$

we see that the Lorentz invariant generalization is

$$P = -\frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^2c^3}\left(\frac{dP_\mu}{d\tau}.\frac{dP^\mu}{d\tau}\right) \quad (10)$$

Here $\tau$ is the proper time of the particle and $p^\mu$ its four-momentum. To verify that this indeed reduces to the correct nonrelativistic limit, let us evaluate the four-vector scalar product:

$$-\left(\frac{dp_\mu}{d\tau}.\frac{dp^\mu}{d\tau}\right) = \left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 \quad (11)$$

From the relativistic relations $E = \gamma mc^2$; $E^2 = p^2c^2 + m^2c^4$; $p = \gamma mv$, we have

$$\frac{dE}{d\tau} = \frac{p}{E}\frac{dp}{d\tau} = \frac{\beta}{c}\frac{dp}{d\tau}$$

so that

$$-\left(\frac{dp_\mu}{d\tau}.\frac{dp^\mu}{d\tau}\right) = \left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2$$

and

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^2c^3}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2\right] \quad (12)$$

In the nonrelativistic limit, $\beta \to 0, \gamma \to 1$, $-\left(\frac{dp_\mu}{d\tau}.\frac{dp^\mu}{d\tau}\right) = \left(\frac{d\vec{p}}{dt}\right)^2$ and the Larmor formula is restored.

The general result (10), or equivalently (12) can be put in an alternative, and often more useful, form. Now

$$\frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt} = \gamma\frac{d(mc\gamma\vec{\beta})}{dt},$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}(1-\beta^2)^{-1/2} = \gamma^3\vec{\beta}.\dot{\vec{\beta}} \Rightarrow \frac{d\vec{p}}{d\tau} = mc[\gamma^4(\vec{\beta}.\dot{\vec{\beta}})\vec{\beta} + \gamma^2\dot{\vec{\beta}}]$$

Similarly

$$\beta \frac{dp}{d\tau} = mc\gamma^4 (\vec{\beta}.\dot{\vec{\beta}})$$

Putting everything together a performing some simple algebra leads to the result

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2] \tag{13}$$

### *27.4. Application to Accelerator Physics*

One area of application of the relativistic expression for radiated power is that of charged particle accelerators. In accelerators, energy is provided to accelerate the particle to high energies to perform scattering and other experiments with the beam of particles. In this context any energy radiated away is an unavoidable loss and sometimes it is the limiting factor on the maximum energy that the beam can achieve. For a given force, i.e., a given rate of change of momentum, the radiative power depends inversely on the square of mass of the particle [equation (12)]. Consequently the radiation losses are the maximum for the electrons,

#### *27.4.1 Linear Accelerators*

Let us look at linear accelerators first. In this case, motion is one dimensional and acceleration is along the direction of velocity so that $\vec{\beta}\times\dot{\vec{\beta}} = 0$. From equation (12)

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2] = \frac{2}{3}\frac{e^2}{m^2c^3}(1-\beta^2)(\frac{dp}{d\tau})^2$$

Or

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}(\frac{dp}{dt})^2 \tag{14}$$

Now from

$$E = \gamma mc^2;\ E^2 = p^2c^2 + m^2c^4;\ p = \gamma mv$$

we get

$$E\frac{dE}{dt} = c^2 p\frac{dp}{dt} \Rightarrow \frac{dp}{dt} = \frac{1}{v}\frac{dE}{dt} = \frac{dE}{dx},$$

In other words, the rate of change of momentum per unit time equals the rate of change of energy per unit distance. Substituting into equation (14), we have

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}(\frac{dE}{dx})^2 \tag{15}$$

This shows that for linear motion the power radiated depends only on the external forces which determine the rate of change of the particle energy with distance and not on the actual energy or momentum of the particle. The ratio of power radiated to power supplied by the source is

$$\frac{P}{(dE/dt)} = \frac{P}{v(dE/dx)} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}\frac{dE}{dx} \to \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2/mc^2}{mc^2}\frac{dE}{dx} \tag{16}$$

where the last form holds for relativistic energies, so that $v \to c$. The loss of energy due to radiation is maximum for the electron. Let us use units that are appropriate for high energy physics. $mc^2$=0.511 MeV is the rest energy of the electron. The quantity $\frac{e^2}{4\pi\varepsilon_0 mc^2}$ has the dimensions of length and is the classical radius of the electron. It has the value $2.82\times10^{-15}$ m. Typical energy gains $(\frac{dE}{dx})$ are of the order of 10 MeV/meter. Thus $\frac{P}{(dE/dt)}$ ~$10^{-15}$ – radiation losses in linear accelerators are completely negligible even for electrons, not to talk of heavier particles like protons and nuclei.

### *27.4.2 Circular Accelerators – Synchrotron Radiation*

Circumstances change dramatically in the case of circular accelerators such as a betatron or a synchrotron. In such accelerators a magnetic field keeps the particle moving in a circular path. At two diagonally opposite points along the circular path the particle is accelerated through an applied potential difference.

*Synchrotron radiation* is the name given to the radiation which occurs when charged particles are accelerated in a curved path or orbit. Classically, any charged particle which moves in a curved path or is accelerated in a straight-line path will emit electromagnetic radiation. Various names are given to this radiation in different contexts. For example, when is occurs upon electron impact with a solid metal target in an x-ray tube, it is called *brehmsstrahlung*.

In the application to circular particle accelerators like *synchrotrons*, where charged particles are accelerated to very high speeds, the radiation is referred to as synchrotron radiation. This radiated energy is proportional to the fourth power of the particle speed and is inversely proportional to the square of the radius of the path. It becomes the limiting factor on the final energy of particles accelerated in electron synchrotrons like the LEP at CERN. However, in other contexts like the *detector arrays* in accelerators, it can be detected and used as an aid to analyzing the products of the scattering event in the accelerator.

Typically in such machines the energy gain per rotation is small but the momentum changes rapidly along the circular path. This means that

$$\left|\frac{d\vec{p}}{d\tau}\right| = \gamma\omega|\vec{p}| >> \frac{1}{c}\frac{dE}{d\tau}$$

Here $\omega$ is the frequency of gyration of the particle in its circular path. As a result the second term in equation (11) is negligible so that

$$-(\frac{dp_\mu}{d\tau}.\frac{dp^\mu}{d\tau}) = \left(\frac{d\vec{p}}{d\tau}\right)^2$$

Using this in equation (10) for the radiated power, gives

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}(\frac{d\vec{p}}{d\tau})^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\omega^2|\vec{p}|^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2c}{\rho^2}\beta^4\gamma^4 \qquad (17)$$

where we have used $\omega = c\beta/\rho$, $\rho$ being the orbit radius. This result was first obtained by Liénard. The radiative energy loss per revolution is

$$\delta E = \frac{2\pi}{\omega}P = \frac{2\pi\rho}{c\beta}P = \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4$$

Now, $E = \gamma mc^2$, and for high energy electrons, $\beta \approx 1$, so that

$$\delta E = \frac{1}{4\pi\varepsilon_0}\frac{4\pi}{3}\frac{e^2}{\rho}(\frac{E}{mc^2})^4 \qquad (18)$$

Substituting for the fundamental constants and choosing units which are appropriate for high energy physics ( $\delta E$ in MeV, energy in GeV and radius in meters), it has the numerical value

$$\delta E(MeV) = 8.85\times 10^{-2}\frac{[E(GeV)]^4}{\rho(meters)} \qquad (19)$$

In the first electron synchrotron built in the 1950's, the radius was around one meter and $E_{max}$ *was around* 0.3GeV. Hence $\delta E_{\max}$ was around one keV per revolution. This was less than, but not negligible, compared to the energy gain of a few keV per turn. In the 10 GeV Cornell electron synchrotron the orbit radius , $\rho$ , was around 100 meters and the radio-frequency voltage per turn was 10.5 MeV at 10 GeV. According to the above estimate, equation 19, the energy loss per turn is 8.85 MeV. Thus at this energy almost the entire power supplied is radiated away. So this put a limit on the maximum energy that could be achieved.

The power radiated in circular electron accelerators is given by equation (17) (where we have put $\beta = 1$)

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2 c}{\rho^2}\gamma^4 \tag{20}$$

The LEP electron synchrotron had a rated energy of 50 GeV and a radius of 4300 meters. This gives $\gamma \approx$ 98,000, and $P \approx 2\times 10^{-7}$ watts per electron. Two-tenths of a microwatt may not sound like much loss, but per electron it is enormous! At this energy the proton velocity would also be essentially c, so the synchrotron radiation loss for the two particles scales like their $\gamma$ factors. At the same energy, the $\gamma$ for a proton would be around 54. So the loss rate for the electron is $(98000/54)^4$ or over $10^{13}$ times the loss for a proton of the same energy in the same synchrotron. In other words, even for circular accelerators radiation losses are significant only for electron machines.

However, synchrotron radiation is not all loss. High energy electrons may be coaxed into emitting extremely bright and coherent beams of high energy photons via synchrotron radiation which have numerous uses in the study of atomic structure, chemistry, condensed matter physics, biology, and technology. Thus there is a great demand for electron accelerators of moderate (GeV) energy and high intensity.

## *27.5 Angular Distribution of Radiation*

As we have seen earlier, for nonrelativistic motion, the radiation emitted by an accelerated charge has a simple $\sin^2\theta$ dependence [equation (7)]. For relativistic motion the acceleration fields depend on velocity in addition to acceleration. As a result the angular distribution is rather complicated. The electric and magnetic fields are given by equations (1) and (2) respectively. Since it is only the acceleration fields that contribute to radiation [second term of equation (1)], from equation (5) the radial component of the Poynting vector can be calculated to be

$$[\vec{S}.\hat{n}]_{ret} = \frac{1}{\mu_0 c}\left|\vec{E}_a\right|^2 = \frac{e^2\varepsilon_0}{4\pi}\left\{\frac{1}{R^2}\left|\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}.\hat{n})^3}\right|^2\right\}_{ret} \tag{21}$$

First we have to be precise as to how we define power radiated. The rate at which radiation crosses a closed surface enclosing the particle depends on that surface due to retardation effects. We are interested in power radiated by the particle rather than the power received at a particular point. In other words we calculate the power not as a function of time *t* at which it is received at the point $\vec{x}$, but as a function of time $t'$ at which it is emitted. Consider the radiation emitted by the charge from time $t' = T_1$ to $t' = T_2$. Consider a surface *S* which encloses the point charge at all times during the emission of radiation. The power emitted at time $t'$ reaches the surface at time $t = t' + \frac{R(t')}{c}$. The power crossing a unit area at $\vec{x}$ on the surface *S* at time *t* is $\vec{S}(\vec{x},t).\hat{n}$, $\hat{n}$ being the unit outward drawn normal. So the total energy crossing this unit area during radiation by the charge is

$$W=\int_{t=T_1+[R(T_1)/c]}^{t=T_2+[R(T_2)/c]}[\vec{S}.\hat{n}]dt=\int_{t'=T_1}^{t'=T_2}[\vec{S}.\hat{n}]\frac{dt}{dt'}dt'$$

We identify the integrand of this expression, $[\vec{S}.\hat{n}]\frac{dt}{dt'}$, as $\frac{dW}{dt'}$, the rate at which the charge radiates what eventually passes a unit area on $S$ at the point $\vec{x}$. This is the *instantaneous* power radiated and hence

$$\frac{dP(t')}{d\Omega}=R^2(\vec{S}.\hat{n})\frac{dt}{dt'}=R^2(\vec{S}.\hat{n})(1-\vec{\beta}.\hat{n}) \tag{22}$$

If the charge is accelerated for a short time during which $\vec{\beta}$ and $\dot{\vec{\beta}}$ are essentially constant in direction and magnitude. Now, far away from the particle, $\hat{n}$ and $\vec{\beta}$ are also constant during the period of acceleration. Then from equation (21) for the Poynting vector, the angular distribution of radiation is given by

$$\frac{dP(t')}{d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\left|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right|^2}{(1-\hat{n}.\vec{\beta})^5} \tag{23}$$

- Integration over the solid angle in this general case is rather involved. However, it can be performed and, as expected, the result is equation (13), the relativistic generalization of Larmor formula.

### *27.5.1 Acceleration Parallel to Velocity*

The simplest example of equation (23) is linear motion, i.e., one in which velocity and acceleration are parallel to each other. In that case $\vec{\beta}\times\dot{\vec{\beta}}=0$. If $\theta$ is the angle between the direction of observation, $\hat{n}$ and $\vec{\beta}$ (or $\dot{\vec{\beta}}$), then equation (23) reduces to

$$\frac{dP(t')}{d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{24}$$

- The total power radiated can be obtained by integrating the above equation over $d\Omega$. An elementary integration yields

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2\dot{v}^2}{c^3}\gamma^6 \tag{25}$$

In agreement with equation (13) with $\vec{\beta}\times\dot{\vec{\beta}} = 0$.

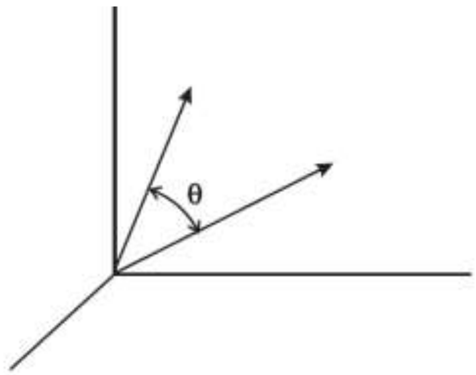
- In the nonrelativistic limit, $\beta$<<1, this reduces to the Larmor result, equation (7).
- But as $\beta$ increases
  - The angular distribution is tipped forward more and more.
  - At the maximum the power increases in magnitude.

- The angle, at which the intensity is maximum, can be obtained by differentiating equation (24) and equating to zero. On putting $\cos\theta = u$; and differentiating with respect to *u*, we have

$$\begin{aligned}
&0 = \frac{d}{du}\frac{1-u^2}{(1-\beta u)^5} \Rightarrow \frac{-2u(1-\beta u)+5\beta(1-u^2)}{(1-\beta u)^6} = 0\\
&\Rightarrow 3\beta u^2 + 2u - 5\beta = 0 \Rightarrow u_{\max} = \frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)\\
&\Rightarrow \theta_{\max} = \cos^{-1}[\frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)]
\end{aligned} \tag{26}$$

- For ultra-relativistic energies, $\beta \to 1$ and the angle $\theta_{\max}$ tends to zero. For finding the limiting value it is easier to substitute for $\beta$ in terms of $\gamma\,(=\frac{1}{\sqrt{1-\beta^2}})$ and take the limit $\gamma \to \infty$. The result is

$$\theta_{\max} \to \frac{1}{2\gamma} \qquad \textbf{[See Figure 14.4 from Jackson Edition 2]} \tag{27}$$




Fig

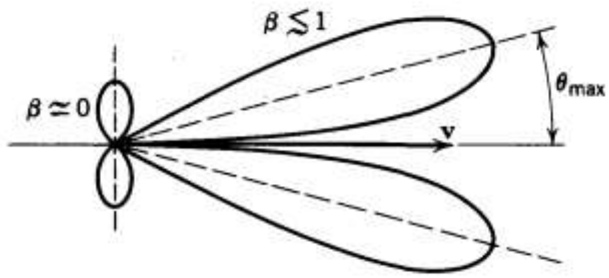



Fig: Radiation pattern for charge accerlated in the direction of motion.

- Thus there is a sharp peak very close to the forward direction, though exactly in the forward direction, the power emitted is zero. However $\theta_{\max}$ can be so small that for all practical purposes the peak is in the forward direction.
- Most of the power is radiated very close to $\theta_{\max}$. For such small angles around $\theta_{\max}$, angular distribution formula (24) can be simplified

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \approx \frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{v}^2}{4\pi c^3}\frac{\theta^2}{[(1-\beta)+\beta\theta^2)]^5}$$

Now put $\beta$ in terms of $\gamma$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ . for $\beta \to 1, (1-\beta) \to \frac{1}{2\gamma}$, so that

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{8e^2\dot{v}^2}{\pi c^3}\gamma^8\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \qquad \textbf{[See Figure]} \qquad (28)$$

- Thus $\gamma^{-1}$ is the natural unit of angle. The peak occurs at the point where the derivative of equation (28) is zero, viz., $\gamma\theta = 1/2$. The half power points can be worked out to be $\gamma\theta = 0.23$ and $\gamma\theta = 0.91$.
- The root mean square angle of emission can be obtained by integrating equation (28):

$$< \theta^2 >= \frac{\int \theta^2 \frac{dP}{d\Omega} d\Omega}{\int \frac{dP}{d\Omega} d\Omega} = \frac{\int \theta^2 \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} d\Omega}{\int \frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} d\Omega} = \frac{1}{\gamma^2}$$

or

$$< \theta^2 >^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E} \qquad (29)$$

### *27.5.2 Acceleration Perpendicular to the velocity*

A particle in instantaneous circular motion has its acceleration perpendicular to its velocity. Let us choose the instantaneous velocity to be in the *z*-direction and acceleration in the *x*-direction, so that $\vec{\beta} = \beta\hat{e}_3$, $\dot{\vec{\beta}} = \dot{\beta}\hat{e}_1$ and $\hat{n} = \sin\theta\cos\phi\hat{e}_1 + \sin\theta\sin\phi\hat{e}_2 + \cos\theta\hat{e}_3$. **[See Figure 14.6 of Jackson Edition 2]** The various vector and scalar products in equation (23) can now be worked out. We have

$$\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}] = (\hat{n}.\dot{\vec{\beta}})(\hat{n} - \vec{\beta}) - (1 - \hat{n}.\vec{\beta})\dot{\vec{\beta}}$$

$$\left|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]\right|^2 = \left|(\hat{n}.\dot{\vec{\beta}})(\hat{n} - \vec{\beta}) - (1 - \hat{n}.\vec{\beta})\dot{\vec{\beta}}\right|^2$$
$$= (\hat{n}.\dot{\vec{\beta}})^2(1 - 2\hat{n}.\vec{\beta} + \beta^2) - 2(\hat{n}.\dot{\vec{\beta}})^2(1 - \hat{n}.\vec{\beta}) + (1 - \hat{n}.\vec{\beta})^2\dot{\beta}^2$$
$$= \sin^2\theta\cos^2\phi(1 - 2\beta\cos\theta + \beta^2)\dot{\beta}^2 - 2\sin^2\theta\cos^2\phi(1 - \beta\cos\theta)\dot{\beta}^2 + (1 - \beta\cos\theta)^2\dot{\beta}^2$$
$$= \dot{\beta}^2[(1 - \beta\cos\theta)^2\beta^2 - (1 - \beta)^2\sin^2\theta\cos^2\phi]$$

On substituting in equation (23), we have

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c^3}\frac{\left|\dot{\vec{v}}\right|^2}{(1 - \beta\cos\theta)^3}[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\cos\theta)^2}] \qquad (30)$$

We observe that

- The total power radiated can be obtained by integrating the above equation over $d\Omega$. An elementary integration yields

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2\dot{v}^2}{c^3}\gamma^4 \tag{31}$$

- In this case there is radiation in the forward direction, $\theta = 0$.
- The detailed angular distribution is rather complicated as two angles are involved.
- However, the same characteristic relativistic peaking in the forward direction is seen.
- Taking the ultra-relativistic limit ($\gamma >> 1$), as we did in the case of linear motion case above, we get

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{\pi c^3}\gamma^6\frac{\dot{v}^2}{(1+\gamma^2\theta^2)^3}[1-\frac{4(\gamma\theta)^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}] \tag{32}$$

- The root mean square angle of emission can be obtained exactly in the same way as before and is

$$<\theta^2>^{1/2} = \frac{1}{\gamma} = \frac{mc^2}{E}, \tag{33}$$

exactly the same as in the earlier case.

### *27.5.3 Comparison of the Two Cases*

From the expressions for the total radiation produced in the two cases, equations (25) and (31), we observe that for a given acceleration the radiation produced in the “parallel” case is a factor $\gamma^2$ more than in the “perpendicular” case. However, this statement is somewhat misleading because what is supplied is the force and not acceleration. A given force produces quite different accelerations depending on whether it is applied along or transverse to the velocity. For a force parallel to the velocity,

$$F = \frac{dp}{dt} = m\gamma^3\frac{dv}{dt}$$

On the other hand, for a force applied normal to velocity

$$F = \frac{dp}{dt} = m\gamma \frac{dv}{dt}$$

Hence, expressed in terms of $\frac{dp}{dt}$, the power in the two cases are

$$P_{\perp} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left(\frac{dp}{dt}\right)^2; \qquad P_{\|} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

Thus for a given force, the radiation produced is a factor $\gamma^2$ *more* when the force is applied perpendicular to the velocity than when it is applied parallel to it. Of course, if the force comes from a magnetic field, it is always perpendicular to the velocity.

***Summary***

1. *The derivation of radiation by moving sources from the electromagnetic fields obtained in the previous module is given.*
2. *The non-relativistic limit of the expression for radiation is obtained leading to the Larmor formula for total emitted radiation.*
3. *Total radiation in the relativistic case is obtained from the unique generalization of the Larmor formula.*
4. *The radiation appears as losses in charged particle accelerators. Such radiation losses are discussed for linear and circular accelerators.*
5. *Angular distribution of radiation in the relativistic case is derived. Two special cases – acceleration parallel to or perpendicular to velocity are discussed. A comparison of the losses in the two cases is made.*